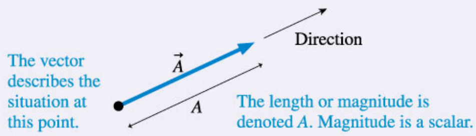
Physics 4A
Chapter 3: Vectors and Coordinate Systems

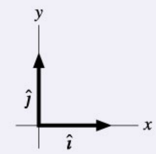
IMPORTANT CONCEPTS

A vector is a quantity described by both a magnitude and a direction.



Unit Vectors

Unit vectors have magnitude 1 and no units. Unit vectors $\hat{\imath}$ and $\hat{\jmath}$ define the directions of the x- and y-axes.



USING VECTORS

Components

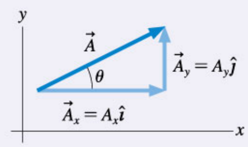
The component vectors are parallel to the x- and y-axes:

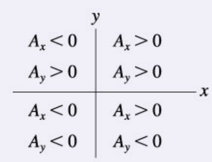
$$\vec{A} = \vec{A}_x + \vec{A}_y = A_x\hat{\imath} + A_y\hat{\jmath}$$

In the figure at the right, for example:

$$A_x = A\cos\theta \qquad A = \sqrt{A_x^2 + A_y^2}$$
$$A_y = A\sin\theta \qquad \theta = \tan^{-1}(A_y/A_x)$$

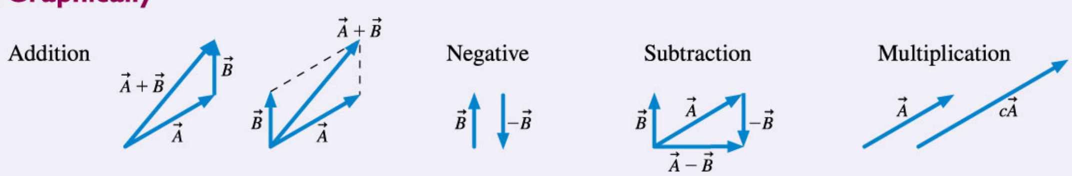
► Minus signs need to be included if the vector points down or left.





The components A_x and A_y are the magnitudes of the component vectors $\vec{A}_x$ and $\vec{A}_y$ *and* a plus or minus sign to show whether the component vector points toward the positive end or the negative end of the axis.

Working Graphically



Working Algebraically

Vector calculations are done component by component: $\vec{C} = 2\vec{A} + \vec{B}$ means $\begin{cases} C_x = 2A_x + B_x \\ C_y = 2A_y + B_y \end{cases}$

The magnitude of $\vec{C}$ is then $C = \sqrt{C_x^2 + C_y^2}$ and its direction is found using $\tan^{-1}$.

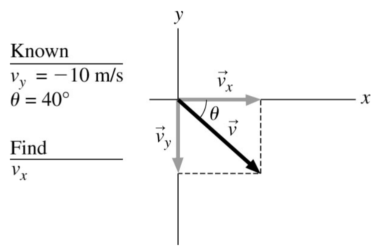
Questions and Example Problems from Chapter 3

Conceptual Question 3.8
Suppose two vectors have unequal magnitudes. Can their sum be zero? Explain.

3.8. No, it is not possible for two vectors with unequal magnitudes to add to zero. To add to zero, two vectors must be antiparallel and of the same length (magnitude).

Problem 3.4
A velocity vector 40° below the positive x-axis has a y-component of -10 m/s. What is the value of its x-component?

3.4. Visualize: The figure shows the components v_x and v_y, and the angle θ.

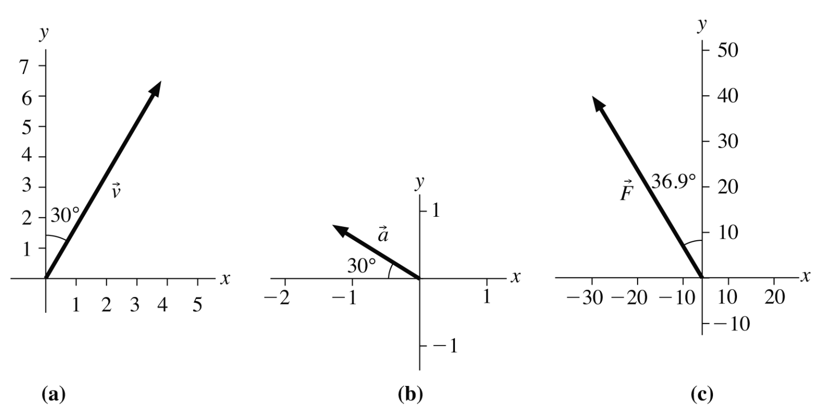


Solve: We have $v_y = -v\sin\theta$ where we have manually inserted the minus sign because $\vec{v}_y$ points in the negative-y direction. The x-component is $v_x = v\cos\theta$. Taking the ratio v_x/v_y and solving for v_x gives $v_x = -v_y(\tan\theta)^{-1} = -(-10\text{ m/s})(\tan 40°)^{-1} = 12\text{ m/s}.$
Assess: The x-component is positive since the position vector is in the fourth quadrant.

Problem 3.7
Draw each of the following vectors. Then find its x- and y-components.

a) $\vec{v}$ = (3.5 m/s, negative x-direction)
b) $\vec{a}$ = (1.5 m/s^2, 30° above the negative x-axis)
c) $\vec{F}$ = (50.0 N, 36.9° counterclockwise from the positive y-axis)

3.7. Visualize:



Solve: **(a)** $v_x = (7.5\text{ m/s})(\sin 30°) = 3.8\text{ m/s};\;\; v_y = (7.5\text{ m/s})(\cos 30°) = 6.5\text{ m/s}$

(b) $a_x = -(1.5\text{ m/s}^2)(\cos 30°) = -1.3\text{ m/s}^2;\;\; a_y = (1.5\text{ m/s}^2)(\sin 30°) = 0.80\text{ m/s}^2$

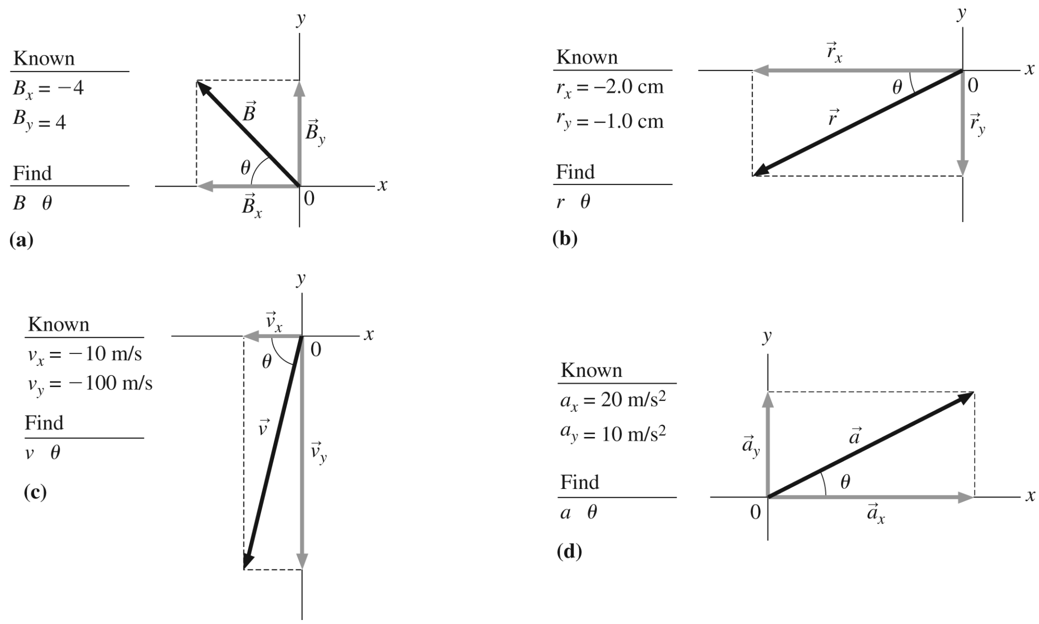
(c) $F_x = -(50.0\text{ N})(\sin 36.9°) = -30\text{ N};\;\; F_y = (50.0\text{ N})(\cos 36.9°) = 40\text{ N}$

Problem 3.10

Draw each of the following vectors, label an angle that specifies the vector's direction, then find its magnitude and direction.

a) $\vec{B} = -4\hat{\imath} + 4\hat{\jmath}$
b) $\vec{r} = (-2.0\hat{\imath} - 1.0\hat{\jmath})$ cm
c) $\vec{v} = (-10\hat{\imath} - 100\hat{\jmath})$ m/s
d) $\vec{a} = (20\hat{\imath} + 10\hat{\jmath})$ m/s^2

3.10. Visualize:



Solve: **(a)** Using the formulas for the magnitude and direction of a vector, we have:

$$B = \sqrt{(-4)^2 + (4)^2} = 5.7,\;\; \theta = \tan^{-1}\left(\frac{4}{4}\right) = 45°$$

(b) $r = \sqrt{(-2.0\text{ cm})^2 + (-1.0\text{ cm})^2} = 2.2\text{ cm},\;\; \theta = \tan^{-1}\left(\frac{1.0}{2.0}\right) = 27°$

(c) $v = \sqrt{(-10\text{ m/s})^2 + (-100\text{ m/s})^2} = 100\text{ m/s},\;\; \theta = \tan^{-1}\left(\frac{100}{10}\right) = 84°$

(d) $a = \sqrt{(10\text{ m/s}^2)^2 + (20\text{ m/s}^2)^2} = 22\text{ m/s}^2,\;\; \theta = \tan^{-1}\left(\frac{10}{20}\right) = 27°$

Problem 3.14

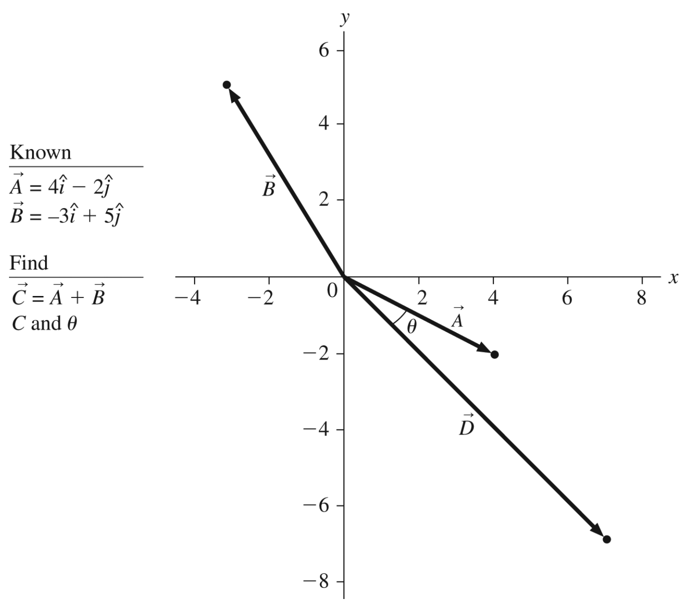
Let $\vec{A} = 4\hat{i} - 2\hat{j},\ \vec{B} = -3\hat{i} + 5\hat{j},\ and\ \vec{D} = \vec{A} - \vec{B}$.

a) Write vector $\vec{D}$ in component form.

b) Draw a coordinate system and on it show vectors $\vec{A},\ \vec{B},\ and\ \vec{D}$.

c) What are the magnitude and direction of vector $\vec{D}$?

3.14. Visualize:



Solve: **(a)** We have $\vec{A} = 4\hat{i} - 2\hat{j},\ \vec{B} = -3\hat{i} + 5\hat{j}$, and $-\vec{B} = 3\hat{i} - 5\hat{j}$. Thus, $\vec{D} = \vec{A} + (-\vec{B}) = (4+3)\hat{i} + (-2-5)\hat{j} = 7\hat{i} - 7\hat{j}$.

(b) Vectors $\vec{A},\ \vec{B}$, and $\vec{D}$ are shown in the figure above.

(c) Since $\vec{D} = 7\hat{i} - 7\hat{j} = D_x\hat{i} + D_y\hat{j},\ D_x = 7$ and $D_y = -7$. Therefore, the magnitude and direction of $\vec{D}$ are

$$D = \sqrt{(7)^2 + (-7)^2} = 7\sqrt{2} = 9.9 \qquad \theta = \tan^{-1}\left(\left|D_y\right|/D_x\right) = \tan^{-1}(7/7) = 45°$$

Assess: Since $\left|D_y\right| = \left|D_x\right|$, the angle $\theta = 45°$, as expected.

Problem 3.16

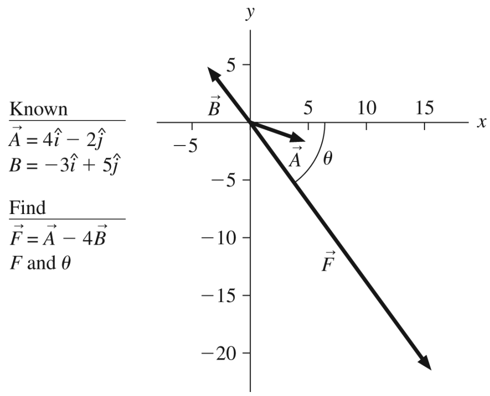
Let $\vec{A} = 4\hat{i} - 2\hat{j},\ \vec{B} = -3\hat{i} + 5\hat{j},\ and\ \vec{F} = \vec{A} - 4\vec{B}$.

a) Write vector $\vec{F}$ in component form.

b) Draw a coordinate system and on it show vectors $\vec{A},\ \vec{B},\ and\ \vec{F}$.

c) What are the magnitude and direction of vector $\vec{F}$?

3.16. Visualize:



Solve: (a) We have $\vec{A} = 4\hat{\imath} - 2\hat{\jmath}$ and $\vec{B} = -3\hat{\imath} + 5\hat{\jmath}$. This means $4\vec{B} = -12\hat{\imath} + 20\hat{\jmath}$. Hence, $\vec{F} = \vec{A} - 4\vec{B} = [4-(-12)]\hat{\imath} + [-2-20]\hat{\jmath} = 16\hat{\imath} - 22\hat{\jmath} = F_x\hat{\imath} + F_y\hat{\jmath}$, so $F_x = 16$ and $F_y = 2\ 22.$

(b) The vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ are shown in the above figure.

(c) The magnitude and direction of $\vec{F}$ are

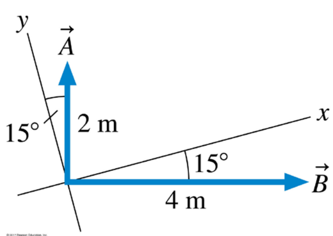
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{(16)^2 + (-22)^2} = 27$$

$$\theta = \tan^{-1}\left(\left|F_y\right|/F_x\right) = \tan^{-1}(22/16) = 54°$$

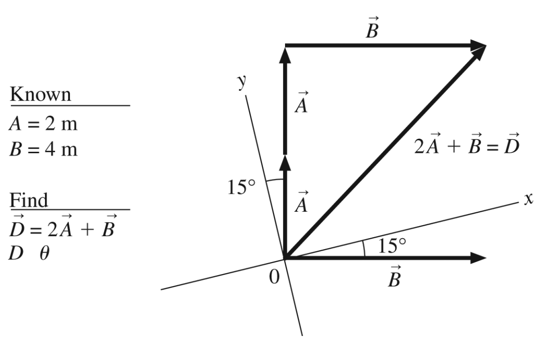
Assess: $F_y > F_x$ implies $\theta > 45°$, which is consistent with the figure.

Problem 3.26

The figure below shows vectors $\vec{A}$ *and* $\vec{B}$. Find $\vec{D} = 2\vec{A} + \vec{B}$. Write your answer in component form.



3.26. Visualize:



Solve: In the tilted coordinate system, the vectors $\vec{A}$ and $\vec{B}$ are expressed as:

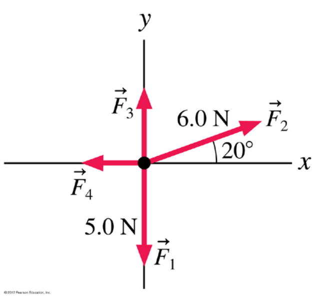
$$\vec{A} = [2\sin(15°)\text{ m}]\hat{i} + [2\cos(15°)\text{ m}]\hat{j} \text{ and } \vec{B} = [4\cos(15°)\text{ m}]\hat{i} - [4\sin(15°)\text{ m}]\hat{j}.$$

Therefore, $\vec{D} = 2\vec{A} + \vec{B} = (4\text{ m})[\sin(15°) + \cos(15°)]\hat{i} + (4\text{ m})[\cos(15°) - \sin(15°)]\hat{j} = (4.9\text{ m})\hat{i} + (2.9\text{ m})\hat{j}.$

Assess: The magnitude of this vector is $D = \sqrt{(4.9\text{ m})^2 + (2.9\text{ m})^2} = 5.7\text{ m},$ and it makes an angle of $\theta = \tan^{-1}(2.9\text{ m}/4.9\text{ m}) = 31°$ with the $+x$-axis. The resultant vector can be obtained graphically by using the rule of tail-to-tip addition.

Problem 3.44

Four forces are exerted on the object shown in the figure below. (Forces are measured in *Newtons*, abbreviated N.) The net force on the object is $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 4.0\hat{i}N$. What are (a) $\vec{F}_3$ and (b) $\vec{F}_4$? Give your answers in component form.



3.44. Visualize: $\vec{F}_3$ and $\vec{F}_4$ are along the axes. The vertical components must add to zero. We treat the horizontal and vertical components separately.

Solve: First find the component vectors of $\vec{F}_2$.

$$\vec{F}_1 = (0.0\text{ N})\hat{i} - (5.0\text{ N})\hat{j} \qquad \vec{F}_2 = (6.0\text{ N})\cos 20°\hat{i} + (6.0\text{ N})\sin 20°\hat{j} = (5.6\text{ N})\hat{i} + (2.1\text{ N})\hat{j}$$

There is no $\hat{i}$ component of $\vec{F}_3$.

$$\vec{F}_3 + (\vec{F}_1)_y + (\vec{F}_2)_y = (0.0\text{ N})\hat{j} \Rightarrow \vec{F}_3 = (5.0\text{ N})\hat{j} - (2.1\text{ N})\hat{j} = (2.9\text{ N})\hat{j}$$

There is no $\hat{j}$ component of $\vec{F}_4$.

$$\vec{F}_4 + (\vec{F}_2)_x = (4.0\text{ N})\hat{i} \Rightarrow \vec{F}_4 = (4.0\text{ N})\hat{i} - (5.6\text{ N})\hat{i} = (-1.6\text{ N})\hat{i}$$

Assess: The figure could have been drawn differently to give negative values for both components; the magnitudes would be the same, however.

Problem A

For the vectors $\vec{a} = (3.0m)\hat{i} + (4.0m)\hat{j}$ and $\vec{b} = (5.0m)\hat{i} + (-2.0m)\hat{j}$, give $\vec{a} + \vec{b}$ in (a) unit-vector notation, and as (b) a magnitude and (c) an angle (relative to $\hat{i}$). Now give $\vec{b} - \vec{a}$ in (d) unit-vector notation, and as (e) magnitude and (f) an angle.

a) $\vec{a}+\vec{b} = [(3.0\text{m})\hat{i}+(4.0\text{m})\hat{j}] + [(5.0\text{m})\hat{i}+(-2.0\text{m})\hat{j}] = \boxed{(8.0\text{m})\hat{i}+(2.0\text{m})\hat{j}}$

b) $\vec{s}=\vec{a}+\vec{b}$: $S_x=(8.0\text{m})$, $S_y=(2.0\text{m})$; $S=\sqrt{S_x^2+S_y^2}=\sqrt{(8.0\text{m})^2+(2.0\text{m})^2}=\boxed{8.2\text{m}}$

c) $\theta=\tan^{-1}(S_y/S_x)=\tan^{-1}\left(\frac{2.0\text{m}}{8.0\text{m}}\right)\longrightarrow\boxed{\theta=14°}$

d) $\vec{d}=\vec{b}-\vec{a}=[(5.0\text{m})\hat{i}+(-2.0\text{m})\hat{j}]-[(3.0\text{m})\hat{i}+(4.0\text{m})\hat{j}]=\boxed{(2.0\text{m})\hat{i}-(6.0\text{m})\hat{j}}$

e) $d=\sqrt{d_x^2+d_y^2}=\sqrt{(2.0\text{m})^2+(-6.0\text{m})^2}=\boxed{6.3\text{m}}$

f) $\theta=\tan^{-1}\left(\frac{-6.0\text{m}}{2.0\text{m}}\right)\longrightarrow\boxed{\theta=-72°}$

check results graphically

Problem B

Find the sum of the following four vectors in (a) unit-vector notation, and as (b) a magnitude and (c) an angle relative to +*x*.

$\vec{P}$: 10.0 m, at 25.0° counterclockwise from +x
$\vec{Q}$: 12.0 m, at 10.0° counterclockwise from +y
$\vec{R}$: 8.00 m, at 20.0° clockwise from -y
$\vec{S}$: 9.00 m, at 40.0° counterclockwise from -y

$\vec{P}=(10.0\text{m})\cos 25.0°\,\hat{i}+(10.0\text{m})\sin 25.0°\,\hat{j}$
$=(9.06\text{m})\hat{i}+(4.23\text{m})\hat{j}$

$\vec{Q}=-(12.0\text{m})\sin 10.0°\,\hat{i}+(12.0\text{m})\cos 10.0°\,\hat{j}$
$=(-2.08\text{m})\hat{i}+(11.8\text{m})\hat{j}$

$\vec{R}=-(8.00\text{m})\sin 20.0°\,\hat{i}-(8.00\text{m})\cos 20.0°\,\hat{j}$
$=(-2.74\text{m})\hat{i}-(7.52\text{m})\hat{j}$

$\vec{S}=(9.00\text{m})\sin 40.0°\,\hat{i}-(9.00\text{m})\cos 40.0°\,\hat{j}$
$=(5.79\text{m})\hat{i}-(6.89\text{m})\hat{j}$

$A_x=P_x+Q_x+R_x+S_x=(9.06\text{m}-2.08\text{m}-2.74\text{m}+5.79\text{m})=10.0\text{m}$

$A_y=P_y+Q_y+R_y+S_y=(4.23\text{m}+11.8\text{m}-7.52\text{m}-6.89\text{m})=1.63\text{m}$

a) $\boxed{\vec{A}=(10.0\text{m})\hat{i}+(1.63\text{m})\hat{j}}$

b) $A=\sqrt{A_x^2+A_y^2}=\sqrt{(10.0\text{m})^2+(1.63\text{m})^2}=\boxed{10.2\text{m}}$

c) $\theta=\tan^{-1}(A_y/A_x)=\tan^{-1}\left(1.63\text{m}/10.0\text{m}\right)\longrightarrow\boxed{\theta=9.25°}$