An Introduction to Data Analysis

Data analysis is the study and evaluation of the uncertainty in a measurement. Experience has shown that no measurement, however carefully made, can be completely free of uncertainty.

In science, the term "error" is used interchangeably with "uncertainty." As such, errors are not mistakes; you cannot avoid them by being very careful. The best you can do is to (a) find reliable estimates of their size and (b) use experimental designs and procedures that keep them as small as possible.

Part 1: Uncertainties in Experimental Measurements

When you report a measurement or experimental result in physics, it's important to always include the uncertainty (as a "plus or minus" amount) as well as the value. For example, when I say that I weigh 150 lbs, I probably don't mean "exactly 150 lbs" but instead "somewhere in the range of 145 to 155 lbs", or 150 ± 5 lbs. Of course, if I were paying close attention to my weight, "150 lbs" might mean somewhere between 149 and 151 lbs (150 ± 1 lbs). Or if I say "a newborn moose weighs about 150 lbs", I might mean somewhere between 100 and 200 lbs (150 ± 50 lbs).

We often use significant figures to imply uncertainty. For example, a result of 12.4 m is usually interpreted as being between somewhere between 12.3 and 12.5 meters. The general understanding is that the implied "plus or minus" amount is at least the size of the last decimal place. However, it's a much better practice to actually state the uncertainty, such as 12.40 ± 0.05 m or 12.4 ± 0.2 m, rather than let the significant figures "imply" an uncertainty. This way, the reader knows that you have thought about the uncertainty rather than just rounding your result to some arbitrary decimal place.

When reporting a value ± uncertainty, the significant figures in the value (and the significant figures in the uncertainty!) should be consistent with the size of the uncertainty. For example, it wouldn't make much sense to report "12.398318 ± 0.05 m", since the uncertainty makes all of the digits after 9 meaningless. It would also be incorrect to report "12 ± 0.05 m", because here the value is not stated with as much precision as the uncertainty allows. Instead, the result should be written as "12.40 ± 0.05 m". Another incorrect statement would be "12.398318 ± 0.046252 m", since the uncertainty itself is uncertain!

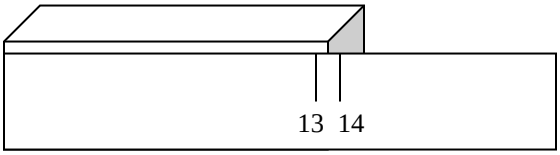
As a general rule, *the uncertainty should be rounded to one significant figure, and the value should be rounded to the same decimal place as the uncertainty* (or one place more, especially if the first digit of the uncertainty is small).

Examples: 112.5 ± 0.5 lbs 124 ± 10 Joules 62,000 ± 5000 km

Since the units of the uncertainty must be the same as the units of the value, it is best to write the units at the end, as shown above.

Estimating uncertainty by "eyeballing"

Sometimes the only way to determine the uncertainty of a measurement is to make an educated guess, or estimate. For example, let's say you are using a ruler to measure the length of a wooden block. You might look at the scale on the ruler and decide that you can read it to the nearest half millimeter (see the figure)



So you would estimate the uncertainty to be 0.5 mm and report that the block's length is 13.5 ± 0.5 mm.

But perhaps the block you're measuring is kind of beat up and doesn't have a nice even edge. In this case the uncertainty arises from the block itself rather than from your ability to read the scale. If you estimate that the block's uneven edge varies by 2 mm, you would report that the block's length is 13.5 ± 2 mm.

Even when you're reading a value from a digital measuring device, such as a stopwatch, estimating uncertainty is important and requires thought. The simplest way is to use the significant figures on the display as a guideline. As discussed above, it is often implied that the uncertainty is half of the smallest decimal place, so a reading of 50.74 seconds would imply an uncertainty of 0.005 seconds, and you might report 50.740 ± 0.005 s.

However, most digital measuring devices have a specified precision (either stamped on the device or listed in a manual), and many of them display more decimal places than their precision implies. For example, the manual for the timer above might state that it is accurate to 0.02 seconds, in which case you should report a value of 50.74 ± 0.02 s.

Estimating uncertainty using statistical analysis

A better way to estimate uncertainty is to make multiple measurements of the same quantity and analyze the data using statistical functions. These are discussed in detail below.

However, it is important to realize that not all types of experimental uncertainties can be assessed by statistical analysis based on repeated measurements. For this reason, uncertainties or errors are classified into two groups: (a) ***random* errors**, which can be treated statistically; and (b) ***systematic* errors**, which cannot. Random errors can be revealed by repeating the measurements; systematic errors cannot.

To illustrate this distinction, let us consider an example. Suppose that we time the period (time for one revolution) of a steadily rotating turntable. One source of error will be our reaction time in starting and stopping the watch. If our reaction time were always exactly the same, these two delays would cancel one another. In practice, however, our reaction time will vary. We may delay more in starting, and so underestimate the time of a

revolution; or we may delay more in stopping, and so overestimate the time. Since either possibility is equally likely, both the size and the sign of the effect are *random*. If we repeat the measurement several times, we will sometimes overestimate and sometimes underestimate. Thus our variable reaction time will show up as a variation in the measured periods. By looking at the spread in the measured periods, we can get a reliable estimate of this kind of error.

On the other hand, if our stopwatch is running consistently slow, then all our times will be underestimates, and no amount of repetition (with the same watch) will reveal this source of error. This kind of error is called *systematic*, because it always pushes our result in the same direction. Systematic errors cannot be discovered by the kind of statistical analysis that we will be discussing below—in fact, they are often hard to evaluate or even to detect. In our introductory lab, we will often (but not always) assume that systematic errors are much smaller than the required precision (and are therefore "negligible").

True values and best estimates

It is important to realize that the "true" period (T) of the rotating turntable discussed above can never be known. Thus, the goal of statistical analysis is to obtain the best estimate (T_{best}) of the true value along with a best estimate of how close T_{best} is likely to be to the true value.

Mean (M): best estimate of the true value

Suppose we make N measurements of a quantity x and get the values $x_1, x_2, ..., x_N$. The best estimate x_{best} of the true value is the ***mean*** or ***average***, defined as:

$$x_{best} = \overline{x} = \frac{\sum x_i}{N}$$

Standard deviation (SD)

The best estimate of the uncertainty in the individual values x_i is the ***standard deviation*** σ_x (or **SD**), defined as:

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(d_i)^2} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

The term d_i in this equation, called the ***deviation,*** is simply the difference between the i$^{\text{th}}$ measurement x_i and the mean value $\overline{x}$. If the deviations are all very small, then our measurements are all close together and are said to be precise.

To be sure we understand the idea of a deviation, let us calculate the deviations for the set of five measurements reported in the table below.

Table 1. Calculation of Deviations

Trial number, i	Measured value, x_i	Deviation, $d_i = x_i - \overline{x}$
1	71	-0.8
2	72	0.2
3	72	0.2
4	73	1.2
5	71	-0.8
	$\overline{x} = 71.8$	$\overline{d} = 0.0$

Notice that some of the deviations are positive and some are negative. In fact, as Table 1 confirms, the average of the deviations is always zero. This is why the standard deviation is found by first squaring the deviations, then averaging these positive squares (**dividing by *N-1* rather than *N***), and finally taking the square root of the result.

For the five measurements in Table 1, the standard deviation σ_x is found to be:

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(d_i)^2} = \sqrt{\frac{1}{5-1}\sum_{i=1}^{5}(x_i - \overline{x})^2} = \sqrt{\frac{1}{4}(0.64 + 0.04 + 0.04 + 1.44 + 0.64)} \approx 0.84$$

Note that this value is bigger than the absolute value of some of the deviations in Table 1 and smaller than others—in other words, it can still be interpreted loosely as the "average" deviation.

Statistical interpretation of the standard deviation

Again, let's suppose that we make *N* measurements of a quantity *x* and get the values $x_1, x_2, ..., x_N$. We then compute the mean, $\overline{x}$, and standard deviation, σ_x. If we then make one more measurement (using the same equipment), statistically there is a 68% probability that the new measurement will fall within one standard deviation of $\overline{x}$ (and a 95% probability that it will fall within two standard deviations of $\overline{x}$). This means that for our new measurement x_{new}, there is a 68% probability that:

$$\overline{x} - \sigma_x < x_{new} < \overline{x} + \sigma_x$$

Now, if the original number of measurements *N* was large, then $\overline{x}$ should be a very reliable estimate for the actual value of *x*. Therefore we can say that there is *a 68% probability that a single measurement will be within standard deviation, σ_x, of the actual value*. Clearly σ_x means exactly what we have used the term "uncertainty" to mean in the preceding sections. If we make one more measurement of *x*, then the uncertainty associated with this measurement can be taken to be σ_x; and with this choice we are 68% confident that our measurement is within σ_x of the correct answer.

In the previous example of Table 1, our best estimate at one standard deviation is:

$$x_{best} = \overline{x} \pm \sigma_x = 71.8 \pm 0.84 \cong 71.8 \pm 0.8$$
$$\Rightarrow 71.0 \le x_{best} \le 72.6$$

If we make one more measurement, one has a 68% confidence level that it will be between 71.0 and 72.6.

From statistical analysis, it can be shown that:

The best estimate x_{best} will fall within the range $\overline{x} \pm \sigma_x$ *approximately* 68% *of the time*
The best estimate x_{best} will fall within the range $\overline{x} \pm 2\sigma_x$ *approximately* 95% *of the time*
The best estimate x_{best} will fall within the range $\overline{x} \pm 3\sigma_x$ *approximately* 99.7% *of the time*

Graphical interpretation of the standard deviation

Suppose that in an experiment we made ten measurements of some length x and obtained the following values (all in cm):

26, 24, 26, 28, 23, 24, 25, 24, 26, 25

A convenient way to organize this data is shown in Table 2. This is known as the ***frequency distribution***.

Table 2

Measured value	23	24	25	26	27	28
Number of measurements	1	3	2	3	0	1

The frequency distribution of our measurements can be graphically displayed in a histogram as shown in Figure 1. Here x_k is the measured length in cm and F_k is the number of times that particular length was measured in our experiment.

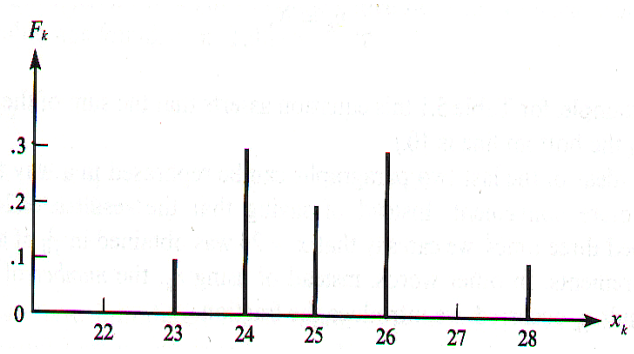


Figure 1

If we could increase the number of measurements (ideally to infinity!), then the histogram would become a bell-shaped curve like those shown in Figure 2.

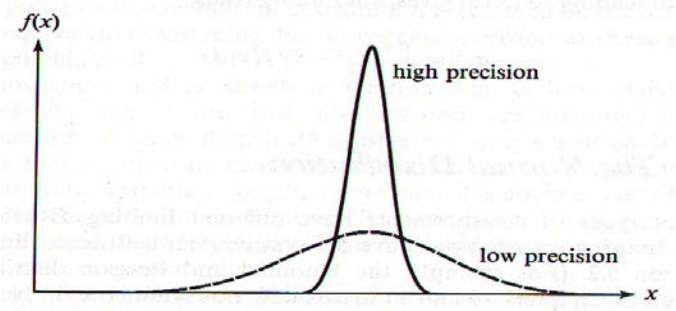


Figure 2

For most types of random errors, the mathematical function that describes this curve is the ***Gaussian*** or "***normal" distribution***:

$$f(x) \propto e^{-(x-\bar{x})^2/2\sigma^2}.$$

Note that this function is characterized by a ***true mean*** $\bar{x}$, which tells us where the peak is, and a ***true standard deviation*** σ, which tells us how wide the curve is. A good approximation is that the width of the curve at ½ the peak height is about 2σ.

The true mean in this function is not the same as the mean of a finite number of measurements, because of the presence of random errors. (Also, the true mean is generally not equal to the true value we are seeking, because of the presence of systematic errors—but it is the best we can hope for using statistics!)